Uniqueness of limit cycles of generalised Lienard systems and predator-prey systems

1988 J. Phys. A: Math. Gen. 21 L685

(http://iopscience.iop.org/0305-4470/21/13/003)

J. Phys. A: Math. Gen. 21 (1988) L685-L691. Printed in the UK

LETTER TO THE EDITOR

Uniqueness of limit cycles of generalised Lienard systems and predator–prey systems

Xun-Cheng Huang

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53233, USA

Received 6 April 1988

Abstract. In this letter, another interesting example of how mathematical methods in physics can be applied to the area of ecology is presented. By using a theorem which first appeared in a Russian paper in 1958 for a generalised Lienard system, a uniqueness theorem of limit cycles of a predator-prey system, which includes Lotka-Volterra, Gause and other systems as special cases, is obtained. Several examples show that the theorem is very useful in dealing with the uniqueness problem of limit cycles for certain ecological systems.

The problem of limit cycles is interesting both in physics and in mathematics. This concept first appeared in the very famous papers by Poincaré (1881, 1882, 1885, 1886). Then in 1926, van der Pol proposed an equation in the study of a self-sustained oscillation occurring in a vacuum tube circuit and showed that the closed orbit in the phase plane of the equation is a limit cycle as considered by Poincaré. After this observation, the existence, non-existence, uniqueness and other properties of limit cycles were studied extensively by mathematicians and physicists. By the 1950s, a lot of mathematical models from physics, engineering, chemistry, biology, economics, etc, were displayed as plane autonomous systems with limit cycles. Therefore, the problem of limit cycles is very important and has attracted the attention of more and more mathematicians, physicists and other scientists. Even in the famous 23 problems of Hilbert, you can find a place for limit cycles (the second part of the sixteenth problem: find the maximum number of limit cycles of all quadratic differential equations).

For the existence and non-existence of limit cycles, there are some old and widely applied results such as the Poincaré-Bendixson theorem, Bendixson criterion and Dulaç criterion (see, for example, Lefschetz 1963). But for the uniqueness problem, the situation is more complicated. While existence or non-existence can be shown by rough estimates or some topological method, the latter needs much more exact estimation.

The uniqueness of limit cycles of the Lienard equation has been proved under certain assumptions by several authors, for example, Lienard (1928), Levinson and Smith (1942) and Sansone (1949). In 1958, Zhang published a paper in Russian and suggested another method to prove the uniqueness of limit cycles of a generalised Lienard system. The complete proof of Zhang's theorem has recently been given by him (Zhang 1986).

In this letter, we propose a general predator-prey system which consists of the Lotka-Volterra system, Gause system, etc as special cases and, by changing variables and utilising Zhang's theorem, we prove the uniqueness of limit cycles for the general

0305-4470/88/130685+07$02.50

predator-prey system. This is, I believe, another good example of how mathematical methods in physics can be applied to the area of ecology. Several examples are given to show that our theorem is very useful in handling the uniqueness problem of limit cycles for certain ecological systems.

The Lienard equation (Lienard 1928)

$$d^2x/dt^2+f(x)\,dx/dt+g(x)=0 \tag{1}$$

can be transferred to the equivalent system

$$\begin{aligned} dx/dt &= -y-F(x) \\ dy/dt &= g(x) \end{aligned} \tag{2}$$

where

$$F(x)=\int_0^x f(x)\,dx \qquad G(x)=\int_0^x g(x)\,dx \tag{3}$$

by the Lienard transformation

$$y=-dx/dt-F(x). \tag{4}$$

Obviously, when $g(x)=x, f(x)=\varepsilon(x^2-1)$, $(\varepsilon>0)$, equation (1) reduces to the van der Pol equation

$$d^2x/dt^2+\varepsilon(x^2-1)\,dx/dt+x=0 \qquad (\varepsilon>0) \tag{5}$$

which has a unique limit cycle for every $\varepsilon>0$.

Zhang (1958) generalised (2) to the following:

$$\begin{aligned} dx/dt &= -\varphi(y)-F(x) \\ dy/dt &= g(x) \end{aligned} \tag{6}$$

and proved (Zhang 1986) the following theorem.

Theorem A. Assume that

(i) $g(x)$ satisfies the Lipschitz condition on every finite interval, $xg(x)>0$, $x\neq 0$, $G(+\infty)=G(-\infty)=+\infty$, where $G(x)=\int_0^x g(x)\,dx$;

(ii) $F'(x)$ is continuous, $F'(x)/g(x)$ is non-decreasing for x in $(-\infty, 0)$ and $(0, +\infty)$ and $F'(x)/g(x)\neq$ constant in a neighbourhood of $x=0$;

(iii) $\varphi(y)$ satisfies the Lipschitz condition on every finite interval, $y\varphi(y)>0$, $y\neq 0$, $\varphi(y)$ is non-decreasing, $\varphi(-\infty)=-\infty$ and $\varphi(+\infty)=+\infty$; $\varphi(y)$ has right and left derivatives at $x=0$ which are non-zero in the case $F'(0)=0$.

Then, system (6) has at most one limit cycles, and if it exists it is stable.

We consider the system

$$\begin{aligned} dx/dt &= \phi(x)(F(x)-\pi(y)) \\ dy/dt &= \rho(y)\psi(x). \end{aligned} \tag{7}$$

Obviously, system (7) consists of the Lotka-Volterra system:

$$\begin{aligned} dx/dt &= x(\alpha-\beta y) \qquad && \alpha, \beta>0 \\ dy/dt &= y(-\gamma+\delta x) \qquad && \gamma, \delta>0 \end{aligned} \tag{8}$$

and Gause-tape systems:

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= xg(x) - yp(x) \\ \mathrm{d}y/\mathrm{d}t &= y(-\gamma + q(x)) \end{aligned} \tag{9}$$

and many other systems (see, for example, Freedman 1980). Also, if we let $\rho(y)$ be constant, $\pi(y) = -\varphi(y)$ and $\phi(x) = -1$, system (7) is reduced to (6). So, in some sense, we can call (7) a more generalised Lienard system.

We want to prove the following uniqueness theorem.

Theorem 1. Suppose that

(i) all the functions in (7) are in $C'(R)$ and $F'(x)$ is in $C(R)$;

(ii) $\phi(0) = \pi(0) = \rho(0) = 0$, $\phi'(x) > 0$, $\psi'(x) > 0$ for $x > 0$, $\rho'(y) > 0$, $\pi'(y) > 0$ for 0 for $y > 0$ and $\pi(+\infty) = +\infty$;

(iii) there exists $x^* > 0$ such that $\psi(x^*) = 0$; also, there exists $k > x^*$ such that $F(k) = 0$ and $(x - k)F(x) < 0$ for $x \neq k$;

(iv) $-F'(x)\phi(x)/\psi(x)$ is non-decreasing for $-\infty < x < x^*$, $x^* < x < +\infty$.

Then, system (7) has at most one limit cycle, and if it exists it is stable.

Proof. Since $\pi(y)$ is strictly increasing, $\pi(0) = 0$, $\pi(+\infty) = +\infty$ and $F(x^*)$ is finite, there exists $y^* > 0$ such that (x^*, y^*) is the unique positive equilibrium point.

By changing variables,

$$\begin{aligned} x &= \xi(u) + x^* \\ y &= \eta(v) + y^* \end{aligned} \tag{10}$$

system (7) can be transferred to

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{\xi'(u)} \phi(\xi(u) + x^*)[F(\xi(u) + x^*) - \pi(\eta(v) + y^*)] \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1}{\eta'(v)} \rho(\eta(v) + y^*)\psi(\xi(u) + x^*). \end{aligned} \tag{11}$$

Let

$$\begin{aligned} \xi'(u) &= \phi(\xi(u) + x^*) \\ \xi(0) &= 0 \end{aligned} \tag{12}$$

and

$$\begin{aligned} \eta'(v) &= \rho(\eta(v) + y^*) \\ \eta(0) &= 0. \end{aligned} \tag{13}$$

Since ϕ and $\rho \in C'(R)$, there do exist ξ and η satisfying (12) and (13). Thus we have

$$\begin{aligned} \mathrm{d}u/\mathrm{d}t &= -[\pi(\eta(v) + y^*) - y^*] - [-F(\xi(u) + x^*) + y^*] = -\varphi(v) - A(u) \\ \mathrm{d}v/\mathrm{d}t &= \psi(\xi(u) + x^*) = g(u). \end{aligned} \tag{14}$$

Now, let us check if the conditions of theorem A are satisfied.

(i) Since $\xi(0) = 0$, $\xi'(u) = \phi(\xi(u) + x^*) > 0$, we have $\xi(u) > 0$ for $u > 0$. Thus, u and $\xi(u)$ have the same sign. Then since $(x - x^*)\psi(x) > 0$,

$$\xi(u)\psi(\xi(u) + x^*) = (\xi(u) + x^* - x^*)\phi(\xi(u) + x^*) > 0$$

and consequently $ug(u) > 0$.

(ii) $$A'(u) = -F'(\xi(u) + x^*)\xi'(u) = -F'(\xi(u) + x^*)\phi(\xi(u) + x^*)$$

is continuous;

$$A'(u)/g(u) = -F'(\xi(u)+x^*)\phi(\xi(u)+x^*)/\psi(\xi(u)+x^*)$$

is non-decreasing for $-\infty < u < 0$, $0 < u < +\infty$; also, since

$$-F'(x)\phi(x)/\psi(x) > 0 \qquad \text{for} \qquad x^* - \varepsilon < x < x^*$$

and

$$-F'(x)\phi(x)/\psi(x) < 0 \qquad \text{for} \qquad x^* < x < x^* + \varepsilon$$

$A'(u)/g(u) \neq$ constant in a neighbourhood of $u = 0$.

(iii) Since $\eta(0) = 0$ and $\eta'(v) = \rho(\eta(v)+y^*) > 0$, $\eta(v) > 0$ for $v > 0$. If $v_1 < v_2$, $\eta'(v_1) - \eta'(v_2) = \rho(\eta(v_1)+y^*) - \rho(\eta(v_2)+y^*) < 0$. Hence $\eta'(v)$ is strictly increasing. Therefore

$$\eta(v) - \eta(0) > \eta'(0)(v-0)$$

which implies

$$\eta(+\infty) = +\infty.$$

So we have

$$v\varphi(v) = v\pi(\eta(v)+y^*) > 0$$

$$\varphi(+\infty) = \pi(\eta(+\infty)+y^*) = \pi(+\infty) = +\infty$$

and $\varphi(v)$, of course, satisfies the other conditions in theorem A.

We thus complete the proof of theorem 1.

Following the criterion of Rosenzweig and MacArthur (1963) and theorem 1, it is easy to prove the next theorem.

Theorem 2. For system (2.7) if, in addition to (i) to (iii),

(v) $F'(x^*) < 0$, then the equilibrium point (x^*, y^*) is stable.

If, in addition to (i) to (iv),

(v)′ $F'(x^*) > 0$, then (x^*, y^*) is unstable and system (7) has a unique limit cycle which is stable.

We now consider the various applications of this method

(*a*) The Rosenzweig-MacArthur system:

$$\begin{aligned} dx/dt &= f(x) - y\phi(x, y) \\ dy/dt &= -ey + ky\phi(x, y) \end{aligned} \tag{15}$$

where $f(x) = ax - bx^2$, $\phi(x, y) = ax/(1+bx)$ and a, b, e, k are positive parameters.

Rosenzweig and MacArthur (1963) studied the stability of system (15) in both graphical and numerical ways, but they found any of their results about limit cycles analytically. Here we just use theorem 2 to show under certain conditions that (15) has a unique stable limit cycle.

Rewrite (15) as

$$\begin{aligned} \frac{dx}{dt} &= \frac{ax}{1+bx}[(a-bx)(1+bx)/a - y] \\ dy/dt &= y[-e + kax/(1+bx)]. \end{aligned} \tag{16}$$

It is not difficult to see, with $\pi(y)=\rho(y)=y$, $\phi(x)=ax/(1+bx)$, $\psi(x)=-e+kax/(1+bx)$ and $F(x)=[a+b(a-1)x-b^2x^2]/a$, that the assumptions in theorem 2 are satisfied. In fact, the unique positive equilibrium point is

$$x^* = e/(ka-eb)$$
$$y^* = 1+eb/(ka-eb)-eb/a(ka-eb)-e^2b^2/a(ka-eb)^2$$

provided that

$$ka-eb-eb/a>0. \tag{17}$$

Since $F'(x)=(ab-b-2b^2x)/a$, if

$$(a-1)(ka-eb)-2eb>0 \tag{18}$$

then

$$F'(x^*)=b[(a-1)(ka-eb)-2eb]/a(ka-eb)>0.$$

Furthermore,

$$W(x)=-F'(x)\phi(x)/\psi(x)=-b[(a-1)x-2bx^2]/[-e+(ka-eb)x]$$

and

$$W'(x)=[2b^2(ka-eb)x^2-4eb^2x^2+eb(a-1)]/[-e+(ka-eb)x]^2>0$$

if (18) is true.

Since (18) implies (17), we have, by theorem 2, the new theorem 3.

Theorem 3. If $(a-1)(ka-eb)-2eb>0$, then the equilibrium point (x^*, y^*) of system (15) is unstable and the system has a unique limit cycle which is stable.

(*b*) A predator-prey system. Kazarinoff and van der Driessche (1978) studied the generalised predator-prey system

$$\begin{aligned} dx/dt &= rx(1-x/k)-yx^n/(a+x^n) \\ dy/dt &= y[\mu x^n/(a+x^n)-D] \end{aligned} \tag{19}$$

where r, k, a, μ and D are positive constants and $\mu>D$, $n\geqslant 1$.

Let

$$(\lambda_n, y_n)=([Da/(\mu-D)]^{1/n}, (r\mu/kD)\,[Da/(\mu-D)]^{1/n}\,\{k-[Da/(\mu-D)]\})$$

which is a equilibrium point of system (19).

Rewriting (19) in the standard form:

$$\begin{aligned} dx/dt &= \frac{x^n}{a+x^n}[rx^{1-n}(1-x/k)(a+x^n)-y] \\ dy/dt &= y[\mu x^n/(a+x^n)-D] \end{aligned} \tag{20}$$

assumptions (i)-(iii) of theorem 1 are obviously satisfied. Since

$$\begin{aligned} F'(x) &= [rx^{1-n}(1-x/k)(a+x^n)]' \\ &= r[(1-n)x^{-n}(1-x/k)(a+x^n)-x^{1-n}(a+x^n)/k+n(1-x/k)] \end{aligned}$$

$$F'(\lambda_n)=(r/kD)[(1-n)(k-\lambda_n)\mu-\lambda_n\mu+nkD-n\lambda_n D] \tag{21}$$

and

$$W(x) = -F'(x)\phi(x)/\psi(x) = -F'(x)x^n/[(\mu - D)x^n - Da]$$
$$= \frac{-r/k}{(\mu - D)x^n - Da}[(1-n)ka + (n-2)ax + kx^n - 2x^{n+1}]$$
$$W'(x) = \frac{r(\mu - D)}{k[(\mu - D)x^n - Da]^2}\{2x^{2n} - [(n-2)(1-n)a + 2(n+1)\lambda_n]x^n$$
$$+ kn[\lambda_n + (1-n)a]x^{n-1} - (2-n)a\lambda_n\}. \qquad (22)$$

According to theorem 2, if

$$F'(\lambda_n) > 0 \text{ and } W'(x) \geq 0 \qquad \text{for} \qquad 0 < x < \lambda_n,\ \lambda_n < x < +\infty$$

then system (19) has a unique limit cycle which is stable.

For example, when $n = 1$, if

$$k > \lambda_1(\mu + D)/D = a + 2\lambda_1 \qquad (23)$$

then

$$F'(\lambda_1) = (r/kD)(kD - \lambda_1\mu - \lambda_1 D) > 0.$$

Also

$$W'(x) = \frac{r(\mu - D)}{k[(\mu - D)x - Da]^2}[2x^2 - 4\lambda_1 x + (k-a)\lambda_1] > 0$$

because

$$16\lambda_1^2 - 8(k-a)\lambda_1 = 8\lambda_1(2\lambda_1 - k + a) < 0.$$

For the case $n = 2$, we have

$$F'(\lambda_2) = (r/kD)[k(2D - \mu) - 2\lambda_2 D]$$

which is positive if $2D - \mu > 0$, $k > 2\lambda_2 D/(2D - \mu)$.

Also

$$W'(x) = \frac{r(\mu - D)}{k[(\mu - D)x^2 - Da]^2}[2x^4 - 6\lambda_2 x^2 + 2k(\lambda_2 - a)x].$$

The discriminant of the trinomial $2x^3 - 6\lambda_2 x + 2k(\lambda_2 - a)$ is positive. In fact

$$\Delta = q^2/4 + p^3/27$$
$$= k^2(\lambda_2 - a)^2/4 + 3^3\lambda_2^3/27$$
$$> 0.$$

This implies that the trinomial has only one real root. Since the product of the three roots is $-k(\lambda_2 - a)$, the real root will be non-positive if $\lambda_2 \geq a$(or if $D \geq a\mu/(1+a)$). Thus

$$W'(x) > 0 \qquad \text{for} \qquad x > 0 \qquad \text{if} \qquad D \geq a\mu/(1+a)$$

and from theorem 2, we have theorem 4.

Theorem 4. For a general predator-prey system (19),

(i) when $n = 1$, it has a unique limit cycle which is stable if $k > a + 2\lambda_1$;

(ii) when $n = 2$, it has a unique limit cycle which is stable if $D \geq a\mu/(1+a)$, $2D - \mu > 0$ and $k > 2\lambda_2 D/(2D - \mu)$.

We can remark that (*a*) the result for $n = 1$ in theorem 4 was obtained by Cheng (1981) but his argument is very tedious and (*b*) theorems 1 and 2 can be applied to many other systems (see, for example, Chen and Jing (1984) and Huang (1987)).

The author would like to thank Professor Stephen J Merrill for valuable discussions.

References

Chen L S and Jing Z J 1984 *Kexue Tongbao* **9** 521
Cheng K S 1981 *SIAM Math. Anal.* **12** 541
Freedman H I 1980 *Deterministic Mathematical Models in Population Ecology* (New York: Marcel Dekker)
Huang Xun-Cheng 1987 *A simple model simulating an immune response. Preprint*
Kazarinoff N and van der Driessche 1978 *Math. Biosci.* **39** 125
Lefschetz S 1963 *Differential Equations: Geometrical Theory* (New York: Interscience)
Levinson N and Smith O 1942 *Duke Math. J.* **9** 382
Lienard A 1928 *Rev. Gen. Elect.* **23** 901
Poincaré H 1881 *J. Math. Pures Appl.* **7** (3) 375
—— 1882 *J. Math. Pures Appl.* **8** (3) 251
—— 1885 *J. Math. Pures Appl.* **1** (4) 167
—— 1886 *J. Math. Pures Appl.* **2** (4) 151
Rosenzweig M L and MacArthur R H 1963 *Am. Nat.* **47** 209
Sansone G 1949 *Ann. Mat. Pure Appl.* **28** 153
van der Pol B 1926 *Phil. Mag.* **2** 11
Zhang Zhifen 1958 *Dokl. Akad. Nauk USSR* **119** 659
—— 1986 *Appl. Anal.* **23** 77